GROUNDING IN MATHEMATICAL STRUCTURALISM

JOHN WIGGLESWORTH

The relation of ground has been used to describe the metaphysical structure of the world. This relation is thought to have certain structural properties: irreflexivity, asymmetry, transitivity, and well-foundedness. This paper examines a putative case of grounding that serves as a counterexample to almost all of these structural properties. The case in question concerns grounding claims made by mathematical structuralists. We focus on two of these claims: (i) that mathematical objects are grounded in the structure that they belong to, and (ii) that mathematical objects are grounded in the other objects in that structure. These cases of ground are particularly interesting for the claim that the grounding relation is well-founded. If they are taken as genuine cases of ground, as we argue they should be, then they provide cases that involve infinitely descending chains of ground. These chains, however, are bounded from below. So they are non-well-founded in one sense, but well-founded in another.

The paper is structured as follows. In section 1 we describe the relation of ground, discussing the nature of the relata, the structural properties of the relation, and some varieties of the relation. Section 2 focuses on specific cases of ground that appear in mathematical structuralism. These cases involve the identities of mathematical objects, a notion that we articulate in section 3. Section 4 makes the grounding claims in mathematical structuralism more precise and argues that they are true by giving an account of mathematical structures in terms of unlabelled graphs. Section 5 concludes with a discussion of the structural properties of the relations of ground that occur in mathematical structuralism, in particular their well-foundedness.

1. THE RELATION OF GROUND

Proponents of the notion of ground argue that the world has a hierarchical structure. This view suggests a picture according to which the world is divided into different levels. The grounding relation is what gives the world this structure, as it represents the basic metaphysical relation that holds between the different levels. It does this by tracking a notion of metaphysical explanation: lower levels in the hierarchy metaphysically explain higher levels. Accordingly, the lower levels ground the higher levels.

The research leading to this paper was carried out within the research project "Mathematics: Objectivity by representation", which is funded by L'Agence Nationale de la Recherche (ANR) and the Deutsche Forschungsgemeinschaft (DFG). I wish to thank Hannes Leitgeb and Graham Priest for helpful discussions of this material and two anonymous referees for comments that helped to improve this paper.

Given that the notion of ground should be understood as a relation, as we assume it should be, there are debates over what its relata are.[1] The position that we take in this paper is that ground is a relation that holds between facts.[2] For those who are metaphysically averse to facts, alternative accounts of ground take the relata to be propositions or sentences. One may also think that the relata can be taken from any of these categories and mixed accordingly, as, e.g., Ross Cameron (2008) and Jonathan Schaffer (2009) do.

Taking ground to be a relation between facts, there are several recognised varieties of ground, which map onto certain distinctions. One particular distinction is of relevance to the present discussion, that between full ground and partial ground.[3] The distinction is easiest to grasp by way of example. A conjunctive fact $A_1 \wedge A_2$ is grounded in its conjuncts, fact A_1 and fact A_2. The conjunction is *partially* grounded in each conjunct taken on its own. But when the two conjuncts are taken together as a pair, the conjunction is *fully* grounded in this pair. This is not to say that the conjunctive fact is fully grounded in the *set* of facts $\{A_1, A_2\}$. There is no need to invoke set theory here (though you may if you are comfortable doing so). The two conjuncts can be taken together as a plurality of facts.

A full ground for a fact B is a sufficient ground for that fact, i.e., a ground for B such that nothing else is required to provide a ground for B. Accordingly, we take full ground to be a many-one relation, as a single fact may not always be sufficient to provide a full ground. A partial ground captures the idea of one fact contributing to the ground of another. The fact B_1 partially grounds the fact B_2 when B_1, possibly together with some other facts, provides a full ground for B_2. Partial ground is then a binary relation, and on this understanding a (single) full ground is also a partial ground.

The grounding relation, in both its full and partial varieties, is taken to have certain structural properties, including irreflexivity, asymmetry, transitivity, and well-foundedness. Some have questioned whether the grounding relation should have all of these properties. Carrie Jenkins (2011) argues that there are cases of reflexive ground, and Jonathan Schaffer (2012) argues that there are counterexamples to the transitivity of ground. We will not be primarily concerned with irreflexivity and transitivity, though the cases we consider are examples that involve a reflexive grounding relation. We are primarily interested in the assumption that the grounding relation is well-founded. Some have challenged the well-foundedness of ground, and plausible cases of non-well-founded ground have been investigated.[4] In what follows, we discuss a natural case of non-well-founded ground that has been sparsely discussed in the literature. This particular case involves grounding claims made by mathematical structuralists. Interestingly, this case also provides a natural counterexample to the asymmetry of ground.

The difference between the cases of ground in mathematical structuralism and other cases of non-well-founded ground in the literature is that many of these latter

[1]For those who reject the idea that ground should be understood as a relation, one can give an account of ground as a sentential connective. See, e.g., Fine (2012) or Schnieder (2011).
[2]Others who take this position include Audi (2012) and Rosen (2010).
[3]Other distinctions include those between weak and strict ground and between mediate and immediate ground. These will not be relevant in the present discussion (though see fn. 6).
[4]See Bliss (2013, 2014), Dixon (forthcoming), Rabin and Rabern (2015), Trogdon (2013).

cases are developed through thought experiments specifically designed to generate counterexamples to the well-foundedness of ground. In the case of mathematical structuralism, however, the grounding claims involved are made independently of any formal study of the relation of ground. For this reason, the cases involving mathematical structuralism might be seen as more natural examples of non-well-founded ground. These cases are also interesting because they occur in the abstract world, as opposed to the concrete world where many studies of ground are focused.

Before looking at these cases in detail, we note that whether a relation is well-founded depends on the particular notion of well-foundedness that one appeals to. There are several different acceptable ways in which a relation may be considered well-founded, three of which will be relevant in the following discussion: (i) being *finitely grounded*, (ii) being *bounded from below*, and (iii) having a *foundation*.[5]

To understand the differences between these notions of well-foundedness it is helpful to consider chains of ground. For simplicity, we focus on chains of partial ground, though the same notions of well-foundedness can be applied to full ground. We symbolize the claim that fact A_1 partially grounds fact A_2 as $A_1 \prec A_2$. A chain of partial ground is then a collection of facts such that for any two facts, A_1 and A_2, in the collection, either $A_1 \prec A_2, A_2 \prec A_1$, or $A_1 = A_2$. A chain of partial ground then has the form:

$$\ldots A_{-2} \prec A_{-1} \prec A_0 \prec A_1 \prec A_2 \ldots$$

In this case, A_{n-1} partially grounds A_n, which partially grounds A_{n+1}, and so on. One version of well-foundedness says that every chain of partial ground is finitely grounded, where a chain is finitely grounded when it does not contain an infinitely descending chain of partial ground. The progression to the left of the above chain must terminate with an ungrounded fact after finitely many steps for the chain to be finitely grounded. A second version of well-foundedness says that every chain of partial ground must be bounded from below. In this case, for every chain of partial ground, there is some fact, F, such that each fact in the chain is either partially grounded by F or identical to F. Note that the grounding fact need not be part of the chain. A third version of well-foundedness does not necessarily invoke the notion of a chain of ground, but simply requires that there be a set of facts S that are not grounded by any fact in the totality of facts, and such that each fact in this totality is either identical to some fact in S or is partially grounded by some fact in S.

Given a collection of facts that is closed under the relation of ground, one can show that if every chain in the collection is finitely grounded, then every chain is bounded from below, and if every chain is bounded from below, then the collection of facts has a foundation. However, the converse implications do not hold. In the case of mathematical structuralism, we show that there are structures that contain infinitely descending chains of ground. However, these structures are bounded from below. It follows that they have a foundation.

[5]See Dixon (forthcoming) and Rabin and Rabern (2015). In what follows, we make use of the notation and terminology from Rabin and Rabern (2015).

2. Grounding and dependence in mathematical structuralism

Given this understanding of the grounding relation, we turn now to an examination of some particular claims made by mathematical structuralists that we argue are claims involving the notion of ground. The claims that we examine concern grounding relations between facts that involve both mathematical objects (e.g., individual natural numbers) and mathematical structures (e.g., the natural number structure). In what follows, we reserve the term "mathematical entity" to refer to both mathematical objects and mathematical structures.

Mathematical structuralism is roughly the view that mathematics is the study of structure. There are different ways to make this view more precise, ways that can be divided into eliminative and non-eliminative approaches. Eliminativists appeal to a background ontology that does not include abstract structures. One might take the background ontology to be sets, and see structures as realised in the universe of sets (Benacerraf 1965). Alternatively, one can take structures to be models of concrete objects, or possible models of concrete objects (Hellman 1989). Non-eliminativists, on the other hand, accept a background ontology of abstract mathematical structures. The grounding claims that we investigate are those of the non-eliminativist, and in what follows, we use the term 'structuralism' and its cognates to mean non-eliminativist mathematical structuralism.

Structuralists appeal to several kinds of claims that sound a lot like grounding claims. Stewart Shapiro, for example, makes these kinds of claims when describing the version of structuralism that he endorses:

> The number 2 is no more and no less than the second position in the natural number structure; and 6 is the sixth position. Neither of them has any independence from the structure in which they are positions, and as positions in this structure, neither number is independent of the other (2000, p. 258).

This passage appeals to a notion of dependence, a notion that is often associated with the notion of ground. Indeed, dependence and ground are sometimes, though not always, used as synonyms for each other. But it's not obvious that this is Shapiro's intention, or the intention of other structuralists who make similar claims (e.g., Resnik 1982).

The question then is whether these dependence claims entail grounding claims that the structuralist would agree to. Shapiro claims that certain natural numbers are dependent on other natural numbers, and that natural numbers are dependent on the natural number structure. These claims express relations of dependence between mathematical entities. Two varieties of dependence between entities are relevant here: *existence* dependence and *identity* dependence. The most straightforward version of existence dependence appeals to a notion of relative existence across (metaphysically) possible worlds: the existence of x depends on the existence of y iff there is no metaphysically possible world where x exists but y doesn't. Shapiro (2008) rejects this version of existence dependence as unsuitable for the characterization of dependence relations in mathematical structuralism.

> [T]o say that the A's and the B's depend on each other is to say that there could not be A's without B's, and vice versa. Despite a proposal I broached tentatively in Shapiro (2006), it seems that this will not do here, since mathematical objects exist of necessity if they exist at all (or so we shall assume). There is no sense of the natural-number structure existing without its places, nor vice versa for that matter. Nor is there any sense of the number four existing without the number six. Or so say our intuitions, or at least my intuitions (p. 302).

Shapiro's alternative is to propose that the dependence relation between a mathematical object and the structure it belongs to is that of non-mereological constitution (2008, p. 303). But the details of this kind of constitution are not provided, nor is it clear whether the account of the dependence of one mathematical object on the others in the same structure is also one of non-mereological constitution. So there is more work to be done on this proposal.

The standard alternative to existence dependence is identity dependence, where one entity depends on another for its identity. For the structuralist, this amounts to the view that the identity of each natural number (for example) depends on the identity of every other natural number in the same structure, and on the identity of the natural number structure itself. This account of dependence in structuralism is most prominently defended by Øystein Linnebo (2008). It is also the view we will develop in the next section, though in a way that differs from Linnebo in order to address certain issues discussed below.

Linnebo argues that mathematical objects depend on other objects in the same structure, and on the structure itself, by appealing to what he calls a weak notion of dependence. According to this notion of dependence, x weakly depends on y iff any individuation of x must make use of entities which also suffice to individuate y.[6] An individuation for Linnebo is an explanation of the identity of an entity. The dependence involved thus qualifies as a form of identity dependence.

Furthermore, if the explanation involved is a metaphysical explanation, then this form of dependence also qualifies as a form of grounding. We argue that the structuralist case involves the notion of metaphysical explanation that is relevant to the grounding relation. Consider Kit Fine (2001) on the connection between grounding and explanation. "We take ground to be an explanatory relation: if the truth that P is grounded in other truths, then they account for its truth; P's being the case holds in virtue of the other truths' being the case" (p. 15). Similarly, Jon Litland (2013) argues that grounding should be understood as *explanation how*. "[G]rounding corresponds to (metaphysical) 'explanation how' in the following sense: when ϕ is grounded in ψ then ψ is a *way* for it to be the case that ϕ. ... What's in question is constitutive explanation: if ψ grounds ϕ then its being the case that ϕ consists in its being the case that ψ" (pp. 19-20). Litland argues that this form of (metaphysical) explanation has a "reasonable claim" to be an appropriate articulation of the notion of ground.

[6] Linnebo's notion of weak dependence is not the same as the notion of weak ground that is often seen in the literature, according to which a fact may be a weak ground for itself. See, e.g., Fine (2012).

On this understanding of ground as an explanatory relation, we argue that the structuralist cases qualify as legitimate cases of ground. A mathematical structure having the particular identity that it has (we will explore what this really amounts to in sections 3 and 4) provides a way, perhaps the only way, for an object in that structure to have the identity it has. Or, in Fine's terms, the object has its identity in virtue of the structure having its identity. Similarly, a mathematical object has its identity in virtue of every other mathematical object in the same structure having the identities they have.

In fact, the connections between objects and structures that these grounding claims make give reasonable interpretations of structuralism's distinguishing theses about the identity of mathematical entities. Independent of Linnebo's account of dependence/grounding in structuralism, arguably many structuralists would agree that the identity of a mathematical object derives from the mathematical structure that it belongs to. For structuralists, the structure is what provides the metaphysical explanation of the objects in the structure. And as every other number in the structure belongs to the structure as well, the identity of each object is explained, at least in part by these other objects and how they are all related. At the very least, there is nothing *outside* of the structure that explains the identity of the objects inside the structure. So if there is any metaphysical explanation of their identities, it must come from the structure itself. Of course the full nature of ground is still being explored, and there are as yet no adequate necessary and sufficient conditions to appeal to. But the connection between grounding and metaphysical explanation tells us that these are cases that involve grounding.[7]

To show that mathematical objects in a structure weakly depend on the other objects in that structure, and on the structure itself, Linnebo starts with a system that comprises a domain D of (not necessarily mathematical) objects, and relations $R_1, \ldots, R_n$ on D. Take R to be the product relation $R_1 \times \ldots \times R_n$. The product relation R is then a complete representation of the original system. Linnebo denotes the abstract structure of R as $\bar{R}$, and invokes what he calls a Dedekind abstraction principle to provide identity conditions for abstract structures: $\bar{R} = \bar{R}'$ iff $R \cong R'$ (that is, iff R is isomorphic to R').[8] If x is an object in the domain of the system represented by R, then the corresponding position in the abstract structure $\bar{R}$ is denoted by $\tau(x, R)$. Identity conditions for these positions, which according to the structuralist are mathematical objects, are given by another abstraction principle: $\tau(x, R) = \tau(x', R')$ iff $\exists f[f : R \cong R' \wedge f(x) = x']$.[9]

Identity conditions are important for Linnebo's account of weak dependence. The account relies crucially on the notion of individuation, where an individuation is a metaphysical explanation of the identity of an entity. The respective identity conditions are supposed to supply individuations of mathematical structures and objects

[7]In the remainder of this section, however, we follow Linnebo and use the term "dependence".

[8]The abstraction principle must be restricted to avoid paradox. For Linnebo, and for us, it is enough to let relation variables range only over sets and have their isomorphism types be non-sets. See Hazen (1985).

[9]Linnebo (2008), pp. 75-6. The function $f : R \cong R'$ is an isomorphism. Note that Linnebo is only considering *rigid* structures, where a structure is rigid iff the only automorphism on the structure is the identity mapping.

in those structures. With this setup established, Linnebo argues that mathematical objects weakly depend on other mathematical objects in the same structure.

> [A]n abstract office [i.e., a position in an abstract structure] can be individuated via an ordered pair $\langle x, R \rangle$, where R is some particular system that realizes the abstract structure S, and where x is an object in the field of R. As part of the system R we are also given occupants of all the other R-offices. We thus have available the entities needed to individuate any of the other abstract offices. This means that each office weakly depends on all the others (2008, p. 79).

Linnebo's notion of weak dependence is interesting, and one that, as Linnebo points out, has received little attention in the literature. However, this notion of dependence is problematic for some non-eliminativist structuralists. Recall the definition of weak dependence: x weakly depends on y iff any individuation of x must make use of entities which also suffice to individuate y, where an individuation of an entity is an explanation of its identity. According to this definition, the individuation of a mathematical object in an abstract structure *must* make use of a particular system R that exhibits that structure, as well as an object x in the domain of R. How is this "must" to be understood? Either it means that this particular system R and this particular object x are crucial, in which case the condition fails, because any other system that exhibits this structure would suffice. Or the claim is weaker, requiring only that some system or other that exhibits the appropriate structure is required. This weaker position may in fact be the one that Linnebo endorses. But if it is, the position commits him to a version of non-eliminativist structuralism called *in re* structuralism.

In re structuralism is often contrasted with *ante rem* structuralism. There are different interpretations of this distinction in the literature. On a coarse-grained interpretation, this distinction is simply the same as the eliminativist (*in re*) and non-eliminativist (*ante rem*) distinction. But on a more fine-grained interpretation, *in re* and *ante rem* structuralists are both non-eliminativist structuralists that appeal to a background ontology of structures. The difference between them lies in the relationship between a structure and the systems or realizations that exemplify that structure. According to *in re* structuralism, the existence of an abstract structure depends on the existence of some system or other that exhibits that structure. Without some realization of the structure, the structure would fail to exist. *Ante rem* structuralists deny this dependence, such that the existence of a structure is independent of the existence of any system or realization that exemplifies the structure. On this view, a structure can exist without any exemplifications.

Linnebo's commitment to *in re* structuralism, on the fine-grained interpretation, is consistent with his account of dependence. In fact the core thesis of *in re* structuralism is a dependence claim: for all abstract structures S, the existence of S depends on the existence of some system exemplifying the structure S.[10] This dependence thesis differentiates *in re* structuralism from *ante rem* structuralism, according to

[10] We mention in passing that this is another interesting claim made by some mathematical structuralists that sounds a lot like a claim involving the notion of ground. We leave exploration of this connection between grounding and structuralism for another occasion.

which no such dependence between structures and systems holds. *Ante rem* structuralism argues that an abstract structure exists independently of any system that exemplifies the structure, and could exist without any such system existing.

Linnebo's account of dependence appears to commit him to a version of *in re* structuralism, which can be seen as follows. In addition to the weak dependence relation between mathematical objects in a structure, Linnebo argues for a weak dependence relation between mathematical objects and the structure they belong to.

> It can also be shown that every office of an abstract structure weakly depends on the structure itself. For in order to individuate such an office we need a realization of the structure. But this is also all we need to individuate the relevant abstract structure itself. (For completeness, I remark that the converse also holds. For the realization needed to individuate the abstract structure contains all the entities needed to individuate its abstract offices as well.) (2008, p. 79)

Accordingly, mathematical objects weakly depend on the structure they belong to. In addition to his weak notion of dependence, Linnebo describes a strong version of dependence, such that x strongly depends on y iff any individuation of x must proceed via y. Returning to Linnebo's argument above, to individuate a mathematical object, we need a system or realization. But the realization is all that's needed to individuate the structure. In other words, any individuation of the structure must proceed via a realization. According to the condition for strong dependence, it follows that a mathematical structure strongly depends on the existence of some system that exemplifies the structure, which is precisely the dependence thesis that *in re* structuralism endorses and *ante rem* structuralism rejects.

In itself the commitment to *in re* structuralism is not problematic. What is problematic is that the account of identity dependence, and so the account of ground related to identity, that Linnebo provides is off limits to *ante rem* structuralists (e.g., Stewart Shapiro) who endorse the dependence of an object the structure it belongs to and on the other objects in that structure. In the next two sections, we provide an alternative account that is available for both *ante rem* and *in re* structuralists to use in order to evaluate grounding claims involving mathematical entities.

3. Identity in Mathematical Structuralism

On our view, grounding is a relation that holds between facts. We argue that structuralists are committed to grounding relations that hold between facts involving the identities of mathematical entities. In terms of facts, the claim that the identity of one entity grounds the identity of another amounts to something like: the fact that one entity has the identity it has grounds the fact that another entity has the identity it has. The crucial notion then is that of having a particular identity.

In a structuralist context, mathematical objects in a structure are not supposed to have any internal nature. What matters are the relations that an object instantiates in that structure. The identity of an object is then given by the relations that it has to objects in the same structure.

> The essence of a natural number is its *relations* to other natural numbers. ... The number 2, for example, is no more and no less than the second position in the natural number structure; 6 is the sixth position. ... There is no more to the individual numbers "in themselves" than the relations they bear to each other (Shapiro 1997, pp. 72-3, emphasis in original).

This passage can reasonably be interpreted as endorsing the view that there is nothing more to the identity of an object in a structure than its structural relations, the relations it bears to other objects in the same structure. This account of the identity of objects in a structure has generated much discussion and Shapiro (2008) has since backed away from it.[11] The main point of contention is that it seems to have counterexamples. These include any structures that have any non-trivial automorphisms, i.e., isomorphisms from the structure to itself that are not the identity mapping. For example, the field of complex numbers has a non-trivial automorphism that maps each number to its conjugate. This automorphism maps, e.g., i to $-i$, implying that i and $-i$ instantiate the same structural relations. If an object's identity is given solely by these structural relations, then i and $-i$ would have the same identity, i.e., would be identical, on this account. But of course they are not, even for the structuralist. On the structuralist view, they are two distinct positions in the complex number structure.

This objection has been resisted by those who argue that the relations of identity and non-identity are structural relations.[12] On this view, the identity of, or difference between, places in a structure is given by the structure itself. The argument for this claim is made by endorsing a connection between structuralism and graph theory. Graph theory, in particular the theory of unlabelled graphs, is of particular interest to structuralism. Indeed, a structure can be seen as an unlabelled graph. Labelled graphs are graphs that include labels for each node, thus making all nodes distinguishable from one another (by their labels), and effectively giving each node an identity independent of its relations to any other nodes in the graph. Unlabelled nodes drop the label, and so if we ignore any relations between the nodes, all of the nodes are indistinguishable from one another. This is very much how structuralists view mathematical objects. Mathematical objects have their identity strictly in virtue of the relations they bear to the objects in the same structure. But to be consistent with mathematical practice, these must include the relations of identity and non-identity. Otherwise, perfectly legitimate mathematical structures would be ruled out. Consider, for example, what Shapiro (1997) calls the cardinal 2 pattern, consisting of two objects with no relations between them. This structure can be represented by the graph with two nodes and no edges (see figure 1).

<INSERT FIGURE 1 HERE>

Though there are no relations between the two different nodes, there are nevertheless two different nodes in this graph. This is just part of what the graph is, a graph

[11]See Burgess (1999), Hellman (2001), Keränen (2001, 2006), MacBride (2006b), Shapiro (1997, 2006, 2008). One thing to come out of the debate over structural relations is the recognition that a precise explication of the notion of a structural relation is needed. No satisfactory account has been given. But the discussion in what follows does not rely on any specific account of structural relations.

[12]See Leitgeb and Ladyman (2008).

FIGURE 1. The two node, edgeless graph (cardinal 2 pattern)

with two distinct nodes. They may be interchangeable; you can permute them and are left with the same graph. But you cannot collapse them into one node. Doing so would result in a different graph. The two node edgeless graph is a perfectly legitimate mathematical entity. And if we associate mathematical structures with graphs, then the structure that this graph represents, the structure with two objects and no relations, is a perfectly legitimate structure. The fact that the two objects are distinct is given by the structure itself.

The upshot is that we can give an account of the identity of an object in a mathematical structure by appealing only to the structural relations of that object. And we can do this without the risk that two distinct objects will be given the same identity. The easiest way to give such an account is to associate with each object the set of structural relations that it instantiates. This set will be the identity of the object.

At the beginning of this section, we formulated the kind of grounding claim we are interested in as: the fact that one entity has the identity it has grounds the fact that another entity has the identity it has. In the context of mathematical structuralism, we can now formulate the claim that mathematical objects are grounded in the other objects in the same structure, where the grounding involves facts concerning the identities of the mathematical objects. Given two mathematical objects n_1 and n_2, we denote the collection of structural relations that each object instantiates as E_{n_1} and E_{n_2}, and take these collections to be the identities of the respective mathematical objects. The grounding claim is then: the fact that the identity of n_1 is E_{n_1} grounds the fact that the identity of n_2 is E_{n_2}. We argue in the next section that this grounding claim is true, when the notion of ground involved is partial ground.

Recall that the structuralist makes two grounding claims, one involving mathematical objects, as above, and one involving mathematical objects and structures. The second grounding claim will then involve the identities of mathematical structures, and so we will need some account of these identities. This will also be made more precise in the following section.

4. MATHEMATICAL STRUCTURALISM, GRAPH THEORY, AND GROUNDING

In this section we develop the view that mathematical structures can be understood as unlabelled graphs, and then use this theory to specify and argue for grounding claims involving mathematical entities.[13] A graph is usually represented set-theoretically as an ordered pair $G = \langle N, E \rangle$ where N is a non-empty collection of nodes, and E is a collection of edges that connect elements of N. If we take

[13]The theory of structures as unlabelled graphs, which can be generalized to multigraphs and hypergraphs, has been articulated by Hannes Leitgeb (2014). The account of grounding developed within this theory here, with whatever faults it may contain, is due to the author.

graphs to represent mathematical structures, then the nodes represent positions in the structure, i.e., mathematical objects, and the edges represent relations between mathematical objects in the structure. The edges of a graph can be represented set-theoretically as well. If the relations in the structure are symmetric, then E is a collection of unordered pairs. But as we usually do not require relations to be symmetric, we can take E to be a collection of ordered pairs. However, all of the set-theoretic representation we have used to describe graphs is merely an explanatory convenience. The theory of unlabelled graphs can be developed by taking the predicates '$Graph(G)$', '$Node(n, G)$', and '$Connected(n_1, n_2, G)$' to be primitive.

An axiomatic theory of graphs based on these primitive predicates can be developed in a second-order language. The full details of this theory are not completely relevant for the present purposes, but here are some of the highlights. In a structuralist context we have the following natural identity conditions for first-order objects of the theory. Two first-order objects, x and y, are identical iff they instantiate exactly the same properties and relations: $x = y \leftrightarrow \forall X[X(x) \leftrightarrow X(y)]$. Importantly, in this theory of unlabelled graphs, first-order objects include both graphs and nodes in graphs. Second-order objects include properties and relations of these first-order objects.

These identity conditions are obviously controversial, especially given the discussion in section 3 and in the work referenced there. All we really need for our purposes is the less controversial left-to-right direction of the biconditional: $x = y \rightarrow \forall X[X(x) \leftrightarrow X(y)]$. If two first-order objects differ with respect to the properties or relations they instantiate, then they are not identical. For nodes in a graph, i.e., mathematical objects, this will manifest itself in a difference with respect to the objects' identities as defined above. The condition ensures that if you change the identity of an object, as we will do when arguing for the relevant grounding claims, then what results is a different mathematical object.

We also have additional identity conditions for graphs, i.e., mathematical structures. Two graphs, G and G', are identical iff they are isomorphic: $G = G' \leftrightarrow G \cong G'$ (the relation $\cong$ can be defined in the theory of graphs with the primitive predicates mentioned above). This is a standard definition of identity for structures in mathematical structuralism.

> [W]e stipulate that two structures are identical if they are isomorphic. There is little need to keep multiple isomorphic copies of the same structure in our structure ontology, even if we have lots of systems that exemplify each one (Shapiro 1997, p. 93).

This definition of identity differentiates the current axiomatic theory of graphs from the usual set-theoretic interpretation. Two ordered pairs $\langle N, E\rangle$ and $\langle N', E'\rangle$ might be set-theoretically distinct, e.g., because they may have different sets of nodes, yet still be isomorphic.

Identifying structures that are isomorphic suggests an account of the identity of a structure. We equate the identity of a structure with its isomorphism class. Given the account of identity between graphs, this will amount to the singleton of the graph itself. This will do for our purposes below, where we use this account of the identity of a structure to discuss grounding relations between mathematical objects and the structures they belong to.

There are standard operations that one can perform on a graph to generate a different graph. The operations we need are those of adding and removing an edge from a graph. Removing an edge e from a graph G results in the largest subgraph $G - e$ that contains all of the nodes of G and all of the edges of G except e. Adding an edge, on the other hand, between nodes n_1 and n_2 in graph G results in the smallest supergraph of G containing an edge between these nodes (Harary 1969, pp. 11-2). For each node n, we have the set $E_n \subseteq E$ of edges that involve n: $E_n = \{\langle n_1, n_2\rangle \in E \mid n = n_1 \vee n = n_2\}$. Take E_n to be the identity of n. If graphs are understood as mathematical structures, removing or adding edges equates to an object changing the relations it bears to other objects in the same structure, i.e., changing the particular identity it has.

With these tools, we argue for two grounding claims in the context of mathematical structuralism that involve mathematical entities. The first claims that, given two mathematical objects in a structure, the identity of one grounds the identity of the other. As grounding is a relation between facts, we have:

ODO: For any two mathematical objects, n_1 and n_2, in the structure G, the fact that the identity of n_1 is E_{n_1} partially grounds the fact that the identity of n_2 is E_{n_2}.

ODO is a claim of partial ground.[14] The fact that the identity of n_1 is E_{n_1}, together with other facts Γ, fully ground the fact that the identity of n_2 is E_{n_2}. The set Γ contains facts about the identities of all of the other objects in G, as the identity of n_2 is partially grounded in these identities too (that is, the choice of n_1 as a partial ground of n_2 in this example is an arbitrary choice).

To evaluate the full grounding claim, we observe that the relation of full ground is usually taken to entail a corresponding necessity claim: if the facts in Δ fully ground the fact that F, then it is necessary that if all of the facts in Δ obtain, then F obtains.[15] Applied to ODO we have: it is necessary that if the identity of n_1 is E_{n_1}, and all the facts in Γ obtain, then the identity of n_2 is E_{n_2}. Contrapositively, it is necessary that if the identity of n_2 is not E_{n_2}, then either the identity of n_1 is not E_{n_1} or some fact in Γ fails to obtain. This is what we show.

The notion of necessity here is crucial. Necessity is usually defined as truth in every situation, and the situations involved are usually metaphysical possibilities. This interpretation gives us the notion of metaphysical necessity. But this notion will not do when applied to grounding and mathematical structuralism. In this context,

[14] As formulated, ODO is a claim involving *distributive* ground, such that each object depends on each of the other objects in a structure. There is, however, an alternative formulation that takes ground to be irreducibly plural. According to this view, there are pluralities of facts grounded in further facts, though no single fact in the plurality is grounded in the further facts (see Dasgupta 2014). Applying irreducibly plural grounds to the structuralist case, it may be that the identities of all of the mathematical objects are grounded together as a plurality of facts. Many thanks to an anonymous referee for suggesting this alternative, a full treatment of which we leave for another occasion.

[15] The connection between grounding and necessity has been disputed. See, e.g., Leuenberger (2014) and Skiles (2015).

we have to be able to consider what happens when mathematical objects change their identities, something that could not happen in any metaphysical possibility. So the situations involved must be something else. We are asking what happens when we change a particular structure G so that a node n_2 in that structure changes with respect to its structural relations E_{n_2}. The situations that are relevant are those structures that are just like G, but which modify E_{n_2} by adding or removing edges between n_2 and some other node. To evaluate the necessity claim, we look at graphs that differ from G only in what is required to make it the case that the identity of n_2 is different from E_{n_2}, and we see what follows from this change.

Consider some arbitrary graph G' just like G, but which differs from G only in what is required to make it that the set of edges that involve n_2 is not E_{n_2}. If, at G' either the identity of n_1 is not E_{n_1}, or some fact in Γ fails to hold, then the relevant necessity claim is true, as the choice of G' was arbitrary. Let E'_{n_2} be the set of edges at G' that involve n_2, which is the result of adding or removing at least one edge, e, to/from E_{n_2}. This edge must be between n_2 and some other node. Suppose it's between n_2 and n_1. This would entail that the identity of n_1 is no longer E_{n_1}, but is rather the set of edges that results from adding or removing e to/from E_{n_1}. So the identity of n_1 at G' is no longer E_{n_1}, and we are done. On the other hand, suppose e is between n_2 and some other node, say n_3. The fact that the identity of n_3 at G is E_{n_3} is a fact contained in Γ. But this fact would no longer hold at G'. The identity of n_3 at G' is no longer E_{n_3} because an edge has been added to or removed from the identity of n_3. So some fact in Γ fails to hold in the relevant situation. We have then, that in any relevant situation, if the identity of n_2 is not E_{n_2}, then either the identity of n_1 is not E_{n_1} or some fact in Γ fails to obtain, because the identity of some other node has changed. As n_1, n_2, and G were arbitrary, this holds for any two objects in a structure.

We have argued for the truth of a necessity claim. Does this necessity claim entail a corresponding grounding claim? In broader discussions of ground this is not generally the case. For example, it is (metaphysically) necessary that if Socrates exists, then the number 2 exists. But neither of these facts grounds the other. The strict conditional holds because the consequent holds necessarily. However, we are in a context where we allow metaphysically necessary truths to fail. Whatever structural relations mathematical objects like n_1 or n_2 instantiate (i.e., whatever identities these objects have), presumably they instantiate these relations necessarily. But we are considering situations where we suppose these objects fail to instantiate some of these relations. We could even consider a situation where such objects fail to exist (in the structure), another metaphysically impossible situation. This can be done by performing the graph-theoretic operation of removing a node, a perfectly legitimate mathematical operation.

We are therefore in a context where we allow situations that might be considered metaphysically impossible. However, these situations, given by the space of possible graphs, are governed by the axioms of the relevant graph theory. And so we can make precise claims about what happens when we consider these situations. Furthermore, by evaluating necessity claims in the space of possible graphs, we are no longer threatened by counterexamples such as those involving the existence of Socrates and the number 2. In the relevant situations, the number 2 does not exist in every mathematical structure. So its existence will not trivially be grounded by the existence of any object whatsoever. Socrates, on the other hand, doesn't exist in any mathematical structure. It may seem like the threat of triviality resurfaces here. As Socrates doesn't exist in any situation, his non-existence is grounded in any fact whatsoever. But really this just shows that it is inappropriate to evaluate grounding claims involving the existence of Socrates by appealing to what's true in the space of possible graphs, something most people would agree with. The account of ground given here is proposed only as an account of grounding in mathematical structuralism, and so it is not threatened by these kinds of counterexamples. And as this threat has been removed, there is then reason to think that the necessity claim we have argued for entails, or at the very least gives evidence for, the corresponding grounding claim ODO.

We note that the argument for ODO shows that the identities of *any* two objects in a structure are grounded in one another. This follows from the fact that Γ contains facts about the identities of all objects in the relevant structure. This version of ODO is consistent with other formulations of ODO in the literature (e.g., Linnebo 2008, p. 67). A weaker version of ODO might claim that only the identities of connected objects are grounded in one another, in which case a modified version of the argument is required. However, if one accepts that the non-identity relation is a structural relation, as discussed in section 3, then all distinct objects in a structure are connected by this relation.

Having argued that the identities of mathematical objects are grounded in the identities of other objects in the same structure, we consider the second grounding claim relevant to mathematical structuralism: that the identity of a mathematical object is grounded in the identity of the mathematical structure it belongs to. To evaluate this grounding claim, we need an account of the identity of a structure, just as we needed an account of the identity of a mathematical object. This account will be given by the identity conditions for graphs.

Recall that identity conditions for graphs are given in terms of isomorphisms. Based on these identity conditions, a natural understanding of the identity of a graph is in terms of its isomorphism class, the collection of graphs that are isomorphic to it. Let $\mathbf{G}$ be the isomorphism class of the graph G. We then say that G has the identity $\mathbf{G}$ iff $G \in \mathbf{G}$. That is, we equate the identity of G with its isomorphism class $\mathbf{G}$. As we are taking graphs to be structures, this gives us an account of the identity of a structure. And so we can formulate the relevant grounding claim.

ODS: For any mathematical object, n, in the structure G, the fact that $G \in \mathbf{G}$ fully grounds the fact that the identity of n is E_n.

As before, this grounding claim is usually taken to entail a corresponding necessity claim, where necessity ranges over the space of relevant possible graphs. In the contrapositive form, this comes to: for all graphs G' that are like G except where the identity of n is not E_n, it follows that $G' \notin \mathbf{G}$. Here $\mathbf{G}$ is the isomorphism class of the graph G that we started with (in effect, the "actual" graph).

Consider some arbitrary graph G' just like G but which differs from G only in what is required to make it that the set of edges that involve n is not E_n. If it follows that $G' \notin \mathbf{G}$, then the relevant necessity claim is true, as the choice of G' was arbitrary. Recall that $\mathbf{G}$ is the isomorphism class of the graph G. And isomorphic graphs are identical. So the isomorphism class of G is simply $\{G\}$. It follows that all we have to show is that $G' \neq G$.

Let E'_n be the set of edges in G' that involve n, which is the result of adding or removing at least one edge, e, to/from E_n. This edge must be between n and some other node m. There are two cases. Either there is an edge between n and m that is removed, so that n and m were connected in G but not connected in G'. Or an edge is added between n and m that was not there before, so that n and m were not connected in G but are connected in G'.

Recall the identity conditions for first-order objects, which hold for graphs as well: $x = y \leftrightarrow \forall X[X(x) \leftrightarrow X(y)]$. We only need the left-to-right direction, or more precisely, the contrapositive of the left-to-right direction. In either of the above cases, there is a property that holds of G but fails to hold of G'. That is, either $Connected(n, m, G)$ and $\neg Connected(n, m, G')$, or vice versa. So these graphs are not identical. The necessity claim under consideration is therefore true. As before, this gives us reason to think that the corresponding grounding claim holds as well.

5. Conclusion

We have argued for two grounding claims involving mathematical entities that are relevant to the mathematical structuralist: that the identity of a mathematical object is grounded in the identity of the structure it belongs to, and in the identities of other mathematical objects in that structure. This argument has proceeded by describing mathematical structures in terms of unlabelled graphs. With this account of structure to hand, we present standard identity conditions for objects in a structure and for structures themselves, which allow us to articulate the notion of the identity of a mathematical entity in the context of structuralism. We then interpret grounding claims involving these entities as claims about what happens in the space of possible mathematical structures. This is an interpretation which makes no reference to any particular systems or realizations that exemplify the structures in question. And so, unlike Linnebo's account, it is an account of grounding that is available to both the *ante rem* and *in re* non-eliminativist structuralists. On this interpretation, we argue that the grounding claims are true. Their truth follows from, or is at least evidenced by, the truth of the relevant corresponding necessity claims, claims ranging over the space of possible mathematical structures.

The notion of ground we appeal to is similar to what has been called the modal account of ground, according to which the fact that A fully grounds the fact that B iff necessarily, if A obtains then B obtains. This account of ground, as a general analysis, is widely agreed to fail when the necessity involved is metaphysical necessity. The failure results from counterexamples to the right-to-left direction, as discussed in section 4. But in certain contexts, this account can be successful, given that the necessity operator is adequately interpreted for the context at hand. We argue that mathematical structuralism is an appropriate context for understanding the relevant grounding claims in terms of necessity claims that range over the space of possible graphs, because the kind of counterexamples that arise in the general case do not arise here.

Given this understanding of the grounding relation in the context of mathematical structuralism, we can evaluate the structural properties of this relation. It is transitive, at least in the context of mathematical structuralism, as we have argued that each object in a structure is grounded in every other object. But more interestingly, it also allows for non-well-founded chains of partial ground. That is, there are chains of partial ground that are not finitely grounded.

Consider the natural number structure. The arguments from section 4 show that the identity of each number is partially grounded by the identity of its predecessor (for simplicity we denote the identity of a number by the standard numeral for that number):

$$1 \prec 2 \prec 3 \prec \ldots$$

This chain of partial ground tracks the usual ordering on the natural numbers and is well-founded. But equally, the identity of each number is partially grounded in the identity of its successor, because the identity of each number is partially grounded in the identities of every other number in the structure. So we have:

$$\ldots \prec 3 \prec 2 \prec 1$$

This chain is clearly not finitely grounded, as it progresses infinitely to the left. These examples also give us symmetric cases of ground, as we have that, e.g., $1 \prec 2$ and $2 \prec 1$. These cases in themselves give rise to chains of partial ground that are not finitely grounded:

$$\ldots \prec 1 \prec 2 \prec 1 \prec 2 \ldots$$

The relation is also reflexive, generating further chains of partial ground that are not finitely grounded:

$$\ldots \prec 1 \prec 1 \prec 1 \prec 1 \ldots$$

But what is also interesting about these chains is that they are all bounded from below. Recall, a chain is bounded from below when there is some fact F (not necessarily in the chain) such that each fact in the chain is either partially grounded by F or identical to F.

The chains in question are actually bounded from below in two ways. As the identity of every number is partially grounded in the identity of every other number, each number serves as a lower bound of partial ground for the chain.[16] But there is also a lower bound of full ground for the chain. This lower bound isn't the identity of any particular number in the structure, as the identity of each number only serves as a partial ground for the others. But the identity of the structure itself is a full ground for each member of the chain, and so it is a lower bound of full ground.

[16]This result suggests that the accepted definition of having a lower bound, as given here, is inadequate, or that an alternative definition is worth exploring in the examination of the well-foundedness of ground.

This latter kind of well-foundedness, being bounded from below with a lower bound of full ground, will arguably hold for any structure, as the arguments in section 4 are widely generalizable and not restricted to any particular structure. And so we have that the identity of each object in a structure is fully grounded in the identity of the structure. Even if we have infinitely descending chains of ground involving the identities of the objects within a structure, the identity of the structure itself will serve as a lower bound for each chain. As each chain in the structure is bounded from below, each chain has a foundation, a set of facts, S, such that each fact in the chain is grounded by some fact in S. In the cases relevant to mathematical structuralism, S will simply contain the single relevant fact concerning the identity of the structure. So while mathematical structuralism gives rise to a relation of partial ground that is non-well-founded in that it is not finitely grounded, the relation does appear to be well-founded in two other important senses, that of being bounded from below and having a foundation.

REFERENCES

Audi, P. (2012). Grounding: Toward a theory of the *in-virtue-of* relation. *Journal of Philosophy*, 109:685–711.

Benacerraf, P. (1965). What numbers could not be. *Philosophical Review*, 74:47–73.

Bliss, R. L. (2013). Viciousness and the structure of reality. *Philosophical Studies*, 166:399 – 418.

Bliss, R. L. (2014). Viciousness and circles of ground. *Metaphilosophy*, 45:245–256.

Burgess, J. (1999). Review of Shapiro (1997). *Notre Dame Journal of Formal Logic*, 40:283–291.

Cameron, R. (2008). Turtles all the way down: Regress, priority and fundamentality. *The Philosophical Quarterly*, 58:1 – 14.

Chalmers, D. J., Manley, D., and Wasserman, R., editors (2009). *Metametaphysics: New Essays on the Foundations of Ontology*. Oxford University Press, Oxford.

Correia, F. and Schnieder, B., editors (2012). *Metaphysical Grounding: Understanding the Structure of Reality*. Cambridge University Press, Cambridge.

Dasgupta, S. (2014). On the plurality of grounds. *Philosophers' Imprint*, 14(20):1–28.

Dixon, T. S. (forthcoming). What is the well-foundedness of grounding? *Mind*.

Fine, K. (2001). The question of realism. *Philosophers' Imprint*, 1:1–30.

Fine, K. (2012). Guide to ground. In Correia and Schnieder (2012), pages 37 – 80.

Hale, R. and Hoffman, A. (2010). *Modality: Metaphysics, Logic, and Epistemology*. Oxford University Press, Oxford.

Harary, F. (1969). *Graph Theory*. Addison-Wesley, Reading, MA.

Hazen, A. P. (1985). Review of Wright (1983). *Australasian Journal of Philosophy*, 63:250–254.

Hellman, G. (1989). *Mathematics Without Numbers*. Oxford: Clarendon Press.

Hellman, G. (2001). Three varieties of mathematical structuralism. *Philosophia Mathematica*, 9:184–211.

Hoelte, M., Schnieder, B., and Steinberg, A., editors (2013). *Varieties of Dependence: Ontological Dependence, Grounding, Supervenience, Response-Dependence*. Philosophia Verlag, Munich.

Jenkins, C. (2011). Is metaphysical dependence irreflexive? *The Monist*, 94:267 – 276.

Keränen, J. (2001). The identity problem for realist structuralism. *Philosophia Mathematica*, 9:308–330.

Keränen, J. (2006). The identity problem for realist structuralism II: A reply to shapiro. In MacBride (2006a), pages 146–163.

Leitgeb, H. (2014). On mathematical structuralism. Paper presented at First Symposium on the Foundations of Mathematics, Vienna.

Leitgeb, H. and Ladyman, J. (2008). Criteria of identity and structuralist ontology. *Philosophia Mathematica*, 16:388–396.

Leuenberger, S. (2014). Grounding and necessity. *Inquiry*, 57:151–174.

Linnebo, Ø. (2008). Structuralism and the notion of dependence. *Philosophical Quarterly*, 58:59 – 79.

Litland, J. E. (2013). On some counterexamples to the transitivity of grounding. *Essays in Philosophy*, 14:19–32.

MacBride, F., editor (2006a). *Identity and Modality*. Clarendon Press, Oxford.

MacBride, F. (2006b). What constitutes the numerical diversity of mathematical objects? *Analysis*, 66:63–69.

Rabin, G. O. and Rabern, B. (2015). Well founding grounding grounding. *Journal of Philosophical Logic*, Online First, DOI 10.1007/s10992-015-9376-4.

Resnik, M. D. (1982). Mathematics as a science of patterns: Epistemology. *Noûs*, 16:95–105.

Rosen, G. (2010). Metaphysical dependence: Grounding and reduction. In Hale and Hoffman (2010), pages 109 – 136.

Schaffer, J. (2009). On what grounds what. In Chalmers et al. (2009), pages 347 – 83.

Schaffer, J. (2012). Grounding, transitivty, and contrastivity. In Correia and Schnieder (2012), pages 122 – 138.

Schnieder, B. (2011). A logic for "because". *Review of Symbolic Logic*, 4:445 – 465.

Shapiro, S. (1997). *Philosophy of Mathematics: Structure and Ontology*. Oxford University Press, Oxford.

Shapiro, S. (2000). *Thinking About Mathematics: The Philosophy of Mathematics*. Oxford: Oxford University Press.

Shapiro, S. (2006). Structure and identity. In MacBride (2006a), pages 109 – 145.

Shapiro, S. (2008). Identity, indiscernibility, and *ante rem* structuralism: The tale of i and $-i$. *Philosophia Mathematica*, 16:285 – 309.

Skiles, A. (2015). Against grounding necessitarianism. *Erkenntnis*, 80:717–751.

Trogdon, K. (2013). An introduction to grounding. In Hoelte et al. (2013).

Wright, C. (1983). *Frege's Conception of Numbers as Objects*. Aberdeen University Press, Aberdeen.